

Schwarzschild Solution from WTDiff Gravity

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Abstract

We study classical solutions in the Weyl-transverse (WTDiff) gravity. The WTDiff gravity is invariant under both the local Weyl (conformal) transformation and the volume preserving diffeomorphisms (transverse diffeomorphisms) and is known to be equivalent to general relativity at least at the classical level. In particular, we find that in a general space-time dimension, the Schwarzschild metric is a classical solution in the WTDiff gravity when it is expressed in the Cartesian coordinate system.

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1 Introduction

The theory of general relativity by Einstein represents a wonderful combination of the theory of gravitation and geometry, resulting in great formal beauty and mathematical elegance. Einstein has taken both the general coordinate invariance (diffeomorphism invariance) and the equivalence principle as the fundamental principle of his gravitational theory. With the help of the Riemannian geometry, the only two fundamental principles fix the physical content of general relativity almost completely and provide us with a playground for discussing various cosmological aspects in the universe. Not to mention the recent discovery of gravitational wave [1], we have thus far watched an overwhelming success of Einstein's general relativity both experimentally and theoretically.

Nevertheless, considerable efforts have been made in order to construct its alternative theories from several reasons. This trend may be justified insofar as the unimodular gravity is concerned [2]-[13] since the cosmological constant problem [14], which is one of the most difficult and important problems in modern theoretical physics, might be solved within this class of the gravitational theory.

Among some aspects of the cosmological constant problem, we are mainly interested in the issue of radiative instability of the cosmological constant: the necessity of fine-tuning the value of the cosmological constant every time the higher-order loop corrections are added in perturbation theory. To resolve this problem, unimodular gravity [2]-[13] has been put forward where the vacuum energy and *a fortiori* all potential energy are decoupled from gravity since in the unimodular condition $\sqrt{-g} = 1$, the potential energy cannot couple to gravity at the action level. In this approach, the value of the cosmological constant is not predicted theoretically but fixed by an initial condition.²

However, in quantum field theories the unimodular condition must be properly implemented via the Lagrange multiplier field. Then, radiative corrections modify the Lagrange multiplier field, which corresponds to the cosmological constant in unimodular gravity, thereby rendering its initial value radiatively unstable. To diminish the contribution of the radiative corrections to the cosmological constant, the Weyl symmetry, or equivalently, the local conformal symmetry, could be added to the volume preserving diffeomorphisms, or equivalently, the transverse diffeomorphisms (TDiff) of unimodular gravity [18]-[24]. We will henceforth call such the theory the Weyl-transverse (WTDiff) gravity.

One of the purposes in this article is to study classical solutions in the WTDiff gravity. Even if the WTDiff gravity is equivalent to general relativity at the classical level, as long as we know, nobody has explicitly derived classical solutions within the framework of the WTDiff gravity. In particular, we wish to investigate whether the Schwarzschild metric is included in the classical solutions of the WTDiff gravity. The Schwarzschild solution is of particular importance since it corresponds to the basic one-body problem of classical astronomy. Indeed, many of the reliable experimental verifications of Einstein equations are based

²Recently, we have established a topological model where the Newton's constant is determined by an initial condition [15]-[17].

on the Schwarzschild line element.

This paper is organised as follows: In Section 2, we review the WTDiff gravity. To this aim, we start with the conformally invariant scalar-tensor gravity, and then fix the gauge symmetries by different gauge conditions [24]. One gauge condition leads to general relativity while the other gauge condition produces the WTDiff gravity. This fact shows the classical equivalence between general relativity and the WTDiff gravity even if local symmetries in the both theories are different. Moreover, we derive equations of motion, and check that they are invariant under the Weyl transformation and the TDiff. In Section 3, we solve the equations of motion of the WTDiff gravity in the static and spherically symmetric ansatz. We show that the Schwarzschild metric in the Cartesian coordinate system is in fact a classical solution. The final section is devoted to discussions.

2 The Weyl-transverse (WTDiff) gravity

We will start with the action of the Weyl-transverse (WTDiff) gravity in a class of unimodular gravity in a general n dimensional space-time [20]-[24], which is given by ³

$$\begin{aligned} S &= \int d^n x \mathcal{L} \\ &= \frac{1}{2} \int d^n x |g|^{\frac{1}{n}} \left[R + \frac{(n-1)(n-2)}{4n^2} \frac{1}{|g|^2} g^{\mu\nu} \partial_\mu |g| \partial_\nu |g| \right], \end{aligned} \quad (1)$$

where we have defined as $g = \det g_{\mu\nu} < 0$. This action (1) turns out to be invariant under not the full group of diffeomorphisms (Diff) but only the transverse diffeomorphisms (TDiff). Moreover, it is worthwhile to notice that in spite of the existence of an explicit mass scale (the reduced Planck mass $M_p = 1$ emerges in the overall constant $\frac{1}{2}M_p^{n-2}$ of the action (1)), this action is also invariant under Weyl transformation. Actually, under the Weyl transformation

$$g_{\mu\nu} \rightarrow g'_{\mu\nu} = \Omega^2(x) g_{\mu\nu}, \quad (2)$$

the Lagrangian density in (1) is changed as

$$\mathcal{L}' = \mathcal{L} - (n-1) \partial_\mu \left(|g|^{\frac{1}{n}} g^{\mu\nu} \frac{1}{\Omega} \partial_\nu \Omega \right). \quad (3)$$

In what follows, let us explain how to derive the action (1) by beginning with the conformally invariant scalar-tensor gravity since this derivation makes it possible to clarify the

³We follow notation and conventions by Misner et al.'s textbook [25], for instance, the flat Minkowski metric $\eta_{\mu\nu} = \text{diag}(-, +, +, +)$, the Riemann curvature tensor $R^\mu{}_{\nu\alpha\beta} = \partial_\alpha \Gamma^\mu_{\nu\beta} - \partial_\beta \Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\sigma\alpha} \Gamma^\sigma_{\nu\beta} - \Gamma^\mu_{\sigma\beta} \Gamma^\sigma_{\nu\alpha}$, and the Ricci tensor $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$. The reduced Planck mass is defined as $M_p = \sqrt{\frac{c\hbar}{8\pi G}} = 2.4 \times 10^{18} \text{ GeV}$. Throughout this article, we adopt the reduced Planck units where we set $c = \hbar = M_p = 1$. In this units, all quantities become dimensionless. Finally, note that in the reduced Planck units, the Einstein-Hilbert Lagrangian density takes the form $\mathcal{L}_{EH} = \frac{1}{2} \sqrt{-g} R$.

equivalence between general relativity and the WTDiff gravity and derive equations of motion of the WTDiff gravity in a concise manner. The conformally invariant scalar-tensor gravity in n space-time dimensions takes the form ⁴

$$S = \int d^n x \sqrt{-g} \left[\frac{n-2}{8(n-1)} \varphi^2 R + \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \right], \quad (4)$$

which is invariant under the Weyl transformation (2) of the metric tensor in addition to the ghost-like scalar field φ as

$$\varphi \rightarrow \varphi' = \Omega^{-\frac{n-2}{2}}(x) \varphi. \quad (5)$$

The gauge condition $\varphi = 2\sqrt{\frac{n-1}{n-2}}$ for the Weyl symmetry leads to the well-known Einstein-Hilbert action of general relativity. On the other hand, the gauge condition $\varphi = 2\sqrt{\frac{n-1}{n-2}}|g|^{-\frac{n-2}{4n}}$ for the longitudinal diffeomorphism results in the action (1) of the WTDiff gravity. Thus, the WTDiff gravity is at least classically equivalent to general relativity since the both actions are obtained via the different choices of gauge condition from the same action (4). Here it is worth stressing that the latter gauge condition is not for the Weyl transformation but for the longitudinal diffeomorphism. Actually, it is easy to see that the latter gauge condition is invariant under the Weyl transformation but breaks the longitudinal diffeomorphism.

In this context, it is useful to comment on the transverse diffeomorphisms and the unimodular condition. First, let us start with the diffeomorphism invariance. Under the general coordinate transformation or Diff, the metric tensor transforms as

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x) \equiv J^\alpha_\mu J^\beta_\nu g_{\alpha\beta}(x), \quad (6)$$

where the Jacobian matrix J^α_μ , which is defined as $J^\alpha_\mu = \frac{\partial x^\alpha}{\partial x'^\mu}$, was introduced. Denoting the determinant of the Jacobian matrix as $J = \det J^\alpha_\mu = \det \frac{\partial x^\alpha}{\partial x'^\mu}$, taking the determinant of Eq. (6) gives us

$$g'(x') = J^2(x)g(x). \quad (7)$$

Then, the transverse diffeomorphisms (TDiff), or equivalently, the volume preserving diffeomorphisms, are defined as a subgroup of the full diffeomorphisms such that the determinant of the Jacobian matrix is the unity

$$J(x) = 1. \quad (8)$$

With this condition (8), the volume element is preserved under the Diff, and Eq. (7) shows that $g(x)$ is a dimensionless scalar field. The existence of such the dimensionless scalar field

⁴This conformally invariant gravity theory has a wide application in phenomenology and cosmology [26]-[29].

in the TDiff gravity, which is a gravitational theory based on the TDiff instead of the Diff or the WTDiff, is thought to be a defect since one cannot exclude any terms with the form of a polynomial of $g(x)$ from the action by the fundamental principles of QFT's [24]. In the infinitesimal form of diffeomorphisms $x^\mu \rightarrow x^{\mu'} = x^\mu - \xi^\mu(x)$, the TDiff can be expressed by

$$\partial_\mu \xi^\mu = 0. \quad (9)$$

The unimodular condition is of course a distinct notion from the TDiff, but is closely related to each other. The unimodular condition is defined as

$$g(x) = -1. \quad (10)$$

This condition, together with Eq. (7), implies the TDiff because of (8). (We have assumed $J > 0$.) Also note that the unimodular condition (10) yields the condition such that the variation of the metric tensor is traceless

$$g^{\mu\nu} \delta g_{\mu\nu} = 0. \quad (11)$$

In the case of the conformally invariant scalar-tensor gravity, one can construct a Weyl invariant metric

$$\hat{g}_{\mu\nu} = \left(\frac{1}{2} \sqrt{\frac{n-2}{n-1}} \varphi \right)^{\frac{4}{n-2}} g_{\mu\nu}. \quad (12)$$

In taking the gauge condition $\varphi = 2\sqrt{\frac{n-1}{n-2}} |g|^{-\frac{n-2}{4n}}$, this metric is reduced to the form

$$\hat{g}_{\mu\nu} = |g|^{-\frac{1}{n}} g_{\mu\nu}. \quad (13)$$

The metric tensor (13) satisfies the unimodular condition (10), so that because of the equation (11), the resultant equations of motion stemming from the WTDiff gravity action (1), become the traceless equations as shown shortly.

Armed with the knowledge of the TDiff, Diff and the unimodular condition, we are ready to show explicitly that the action (1) of the WTDiff gravity is indeed invariant under not the Diff but the TDiff. For this purpose, let us perform the Diff in the Lagrangian density of (1) whose result is given by

$$\mathcal{L}'(x') = \frac{1}{2} |J^2 g|^{\frac{1}{n}} \left[R + \frac{(n-1)(n-2)}{4n^2} \frac{1}{|g|^2} g^{\mu\nu} (\partial_\mu |g| + \frac{2|g|}{J} \partial_\mu J) (\partial_\nu |g| + \frac{2|g|}{J} \partial_\nu J) \right]. \quad (14)$$

It is obvious that the Lagrangian density \mathcal{L} is not invariant under the Diff owing to the presence of the terms with J , but when $J = 1$ as in (8) in the case of the TDiff, the Lagrangian density \mathcal{L} becomes invariant, which means that the TDiff are in fact a symmetry of the action (1) of the WTDiff gravity.

Next, we will derive the equations of motion for the WTDiff gravity (1). A method of the derivation is to work with the action (4) of the conformally invariant scalar-tensor gravity, derive its equations of motion, and then substitute the gauge condition $\varphi = 2\sqrt{\frac{n-1}{n-2}}|g|^{-\frac{n-2}{4n}}$ into them. After some calculations, it turns out that the action (4) produces the equations of motion for $g_{\mu\nu}$ and φ , respectively

$$\frac{n-2}{8(n-1)} \left[\varphi^2 G_{\mu\nu} + (g_{\mu\nu} \square - \nabla_\mu \nabla_\nu)(\varphi^2) \right] = \frac{1}{4} g_{\mu\nu} \partial_\rho \varphi \partial^\rho \varphi - \frac{1}{2} \partial_\mu \varphi \partial_\nu \varphi, \quad (15)$$

and

$$\frac{n-2}{4(n-1)} \varphi R = \square \varphi, \quad (16)$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor and $\square \varphi = g^{\mu\nu} \nabla_\mu \nabla_\nu \varphi$. It is well-known that the conformally invariant scalar-tensor gravity can be obtained from the Einstein-Hilbert action via the Weyl-invariant metric $\hat{g}_{\mu\nu} \propto \varphi^{\frac{4}{n-2}} g_{\mu\nu}$, so the equation of motion (16) for the *spurion* field φ should be not independent of the equations of motion (15) for the metric tensor. In fact, taking the trace part of Eq. (15) naturally leads to Eq. (16). Thus, it is sufficient to take only the equations of motion (15) into consideration. Substituting the gauge condition $\varphi = 2\sqrt{\frac{n-1}{n-2}}|g|^{-\frac{n-2}{4n}}$ into Eq. (15) gives us the equations of motion for the WTDiff gravity

$$G_{\mu\nu}^T = \Delta_{\mu\nu}^T, \quad (17)$$

where $G_{\mu\nu}^T \equiv R_{\mu\nu} - \frac{1}{n} g_{\mu\nu} R$ is the *traceless* Einstein tensor and $\Delta_{\mu\nu}^T$ is also a traceless object defined by

$$\begin{aligned} \Delta_{\mu\nu}^T &= \frac{(n-2)(2n-1)}{4n^2} \left[\frac{1}{|g|^2} \partial_\mu |g| \partial_\nu |g| - \frac{1}{n} g_{\mu\nu} \frac{1}{|g|^2} (\partial_\rho |g|)^2 \right] \\ &\quad - \frac{n-2}{2n} \left[\frac{1}{|g|} D_\mu D_\nu |g| - \frac{1}{n} g_{\mu\nu} \frac{1}{|g|} D_\rho D^\rho |g| \right], \end{aligned} \quad (18)$$

where we have defined $D_\mu D_\nu |g| = \partial_\mu \partial_\nu |g| - \Gamma_{\mu\nu}^\rho \partial_\rho |g|$. The explicit existence of g in $\Delta_{\mu\nu}^T$ clearly indicates that the equations of motion for the WTDiff gravity are not invariant under the full Diff. Finally, note that as mentioned before, Eq. (17) is purely a traceless equation.

The equations of motion for the WTDiff gravity, (17), are derived by starting with the action of the conformally invariant scalar-tensor gravity which is invariant under both the Weyl transformation and the Diff, but the gauge condition breaks Diff down to TDiff. Thus, the equations of motion (17) should be invariant under both the Weyl transformation and the TDiff. Let us demonstrate this fact by an explicit calculation.

Under the Weyl transformation (2), the traceless Einstein tensor $G_{\mu\nu}^T$ and $\Delta_{\mu\nu}^T$ are transformed by the same quantity

$$\begin{aligned} G_{\mu\nu}^{T'} &= G_{\mu\nu}^T + A_{\mu\nu}^T, \\ \Delta_{\mu\nu}^{T'} &= \Delta_{\mu\nu}^T + A_{\mu\nu}^T, \end{aligned} \quad (19)$$

where $A_{\mu\nu}^T$ is defined as

$$A_{\mu\nu}^T = 2(n-2)\frac{1}{\Omega^2} \left[\partial_\mu \Omega \partial_\nu \Omega - \frac{1}{n} g_{\mu\nu} (\partial_\rho \Omega)^2 \right] - (n-2)\frac{1}{\Omega} \left[\nabla_\mu \nabla_\nu \Omega - \frac{1}{n} g_{\mu\nu} \nabla_\rho \nabla^\rho \Omega \right]. \quad (20)$$

It is therefore obvious that Eq. (17) is invariant under the Weyl transformation.

Next, let us perform the general coordinate transformation to Eq. (17) whose result is described as

$$\begin{aligned} G_{\mu\nu}^{T'} - \Delta_{\mu\nu}^{T'} &= J_{\mu'}^\alpha J_{\nu'}^\beta \left\{ G_{\alpha\beta}^T - \Delta_{\alpha\beta}^T + \frac{n-2}{2n} \left[\frac{1}{n} \frac{1}{J|g|} (\partial_\alpha J \partial_\beta |g| + \partial_\beta J \partial_\alpha |g|) \right. \right. \\ &+ \frac{2(1-n)}{n} \frac{1}{J^2} \partial_\alpha J \partial_\beta J + \frac{2}{J} D_\alpha D_\beta J \left. \right] - \frac{n-2}{n^2} \left[\frac{1}{n} \frac{1}{J|g|} \partial_\rho J \partial^\rho |g| \right. \\ &+ \left. \left. \frac{1-n}{n} \frac{1}{J^2} (\partial_\rho J)^2 + \frac{1}{J} D_\rho D^\rho J \right] g_{\alpha\beta} \right\}. \end{aligned} \quad (21)$$

From this expression, we see that (17) is not invariant under the Diff, but with $J = 1$, that is, under the TDiff, it becomes invariant. In this way, we have shown that Eq. (17) is invariant under the Weyl transformation as well as the TDiff.

3 Schwarzschild solution

In this section, we wish to show that the Schwarzschild metric is a classical solution to the equations of motion of the WTDiff gravity, (17). Before doing so, let us study a general feature of Eq. (17).

In attempting to analyse a structure of classical solutions to Eq. (17), we soon realize that a notable feature of Eq. (17) is that $G_{\mu\nu}^T = R_{\mu\nu} - \frac{1}{n} g_{\mu\nu} R$ in the LHS has a beautiful geometrical structure whereas $\Delta_{\mu\nu}^T$ in the RHS has a ugly expression, and the presence of the metric determinant g and its derivative $D_\mu D_\nu |g|$ reflects the fact that the equations of motion are not invariant under the Diff, but only the TDiff. In this respect, note that $D_\mu D_\nu |g|$ has a bizarre transformation property. Thus, it is natural to fix the Weyl symmetry first by the gauge condition

$$g = -1, \quad (22)$$

which is nothing but the unimodular condition (10). Since the unimodular condition naturally yields Eq. (8) as mentioned before, this gauge condition does not break the TDiff.

Then, $\Delta_{\mu\nu}^T$ in the RHS of Eq. (17) trivially vanishes so that we have the equations

$$G_{\mu\nu}^T \equiv R_{\mu\nu} - \frac{1}{n} g_{\mu\nu} R = 0. \quad (23)$$

The space-time defined by Eq. (23) is called Einstein spaces in four dimensions and the study of the Riemannian spaces which are conformally related to Einstein spaces, has been addressed for a long time [30]. Together with the Bianchi identity, Eq. (23) leads to

$$\frac{n-2}{2n}\nabla_\mu R = 0, \quad (24)$$

implying the constant curvature spaces except in two dimensions.

Now we are willing to demonstrate that the Schwarzschild metric in the Cartesian coordinate system is a classical solution to the equations of motion of the WTDiff gravity, (17). Since we take the gauge condition (22) for the Weyl transformation, classical solutions in which we are interested belong to a subgroup of Einstein spaces where the gauge condition (22) is imposed as an additional condition.

We wish to look for a gravitational field outside an isolated, static, spherically symmetric object with mass M . In the far region from the isolated object, we assume that the metric tensor is in an asymptotically Lorentzian form

$$g_{\mu\nu} \rightarrow \eta_{\mu\nu} + \mathcal{O}\left(\frac{1}{r^{n-3}}\right), \quad (25)$$

where $\eta_{\mu\nu}$ is the Minkowski metric, and the radial coordinate r is defined as

$$r = \sqrt{(x^1)^2 + (x^2)^2 + \cdots + (x^{n-1})^2} = \sqrt{(x^i)^2}, \quad (26)$$

with i running over spatial coordinates ($i = 1, 2, \dots, n-1$).

Let us recall that the most spherically symmetric line element in n space-time dimensions reads

$$ds^2 = -A(r)dt^2 + B(r)(x^i dx^i)^2 + C(r)(dx^i)^2 + D(r)dt x^i dx^i, \quad (27)$$

where $A(r)$ and $C(r)$ are positive functions depending on only r . Requiring the invariance under the time reversal $t \rightarrow -t$ leads to $D = 0$. As is well-known, we can set $C(r) = 1$ by redefining the radial coordinate r [31]. Thus, the line element under consideration takes the form in the Cartesian coordinate system

$$ds^2 = -A(r)dt^2 + (dx^i)^2 + B(r)(x^i dx^i)^2. \quad (28)$$

From this line element (28), the non-vanishing components of the metric tensor are given by

$$g_{tt} = -A, \quad g_{ij} = \delta_{ij} + Bx^i x^j, \quad (29)$$

and the components of its inverse matrix are

$$g^{tt} = -\frac{1}{A}, \quad g^{ij} = \delta^{ij} - \frac{B}{1 + Br^2}x^i x^j. \quad (30)$$

Moreover, using these components of the metric tensor, the affine connection is calculated to be

$$\begin{aligned}\Gamma_{ti}^t &= \frac{A'}{2A} \frac{x^i}{r}, & \Gamma_{tt}^i &= \frac{A'}{2(1+Br^2)} \frac{x^i}{r}, \\ \Gamma_{jk}^i &= \frac{1}{2(1+Br^2)} \frac{x^i}{r} (2Br\delta_{jk} + B'x^jx^k),\end{aligned}\quad (31)$$

where we have defined $A' = \frac{dA}{dr}$, for instance.

At this stage, let us take the gauge condition (22) for the Weyl transformation. By means of the metric tensor (29), the gauge condition (22) is cast to the form

$$A(1+Br^2) = 1. \quad (32)$$

Using this gauge condition (32) and Eqs. (29)-(31), the Ricci tensor and the scalar curvature can be easily calculated to be

$$\begin{aligned}R_{tt} &= \frac{1}{2}A(A'' + \frac{n-2}{r}A'), \\ R_{ij} &= \left[\frac{n-3}{r^2}(1-A) - \frac{A'}{r} \right] \delta_{ij} + \frac{1}{r^2} \left[\frac{n-3}{r^2}(A-1) + \frac{1}{r} \frac{A'}{A} (1 - \frac{n}{2} + A) - \frac{1}{2} \frac{A''}{A} \right] x^i x^j, \\ R &= -A'' - \frac{2(n-2)}{r}A' - \frac{(n-2)(n-3)}{r^2}(A-1).\end{aligned}\quad (33)$$

These results produce the concrete expressions for the non-vanishing components of the traceless Einstein tensor $G_{\mu\nu}^T \equiv R_{\mu\nu} - \frac{1}{n}g_{\mu\nu}R$

$$\begin{aligned}G_{tt}^T &= \left(\frac{1}{2} - \frac{1}{n} \right) A \left[A'' + (n-4)\frac{1}{r}A' - 2(n-3)\frac{1}{r^2}(A-1) \right], \\ G_{ij}^T &= \left\{ \frac{1}{n}\delta_{ij} + \frac{1}{r^2} \frac{1}{A} \left[-\frac{1}{2} + \frac{1}{n}(1-A) \right] x^i x^j \right\} \left[A'' + (n-4)\frac{1}{r}A' \right. \\ &\quad \left. - 2(n-3)\frac{1}{r^2}(A-1) \right].\end{aligned}\quad (34)$$

As a result, Eq. (23) reduces to the equation

$$A'' + (n-4)\frac{1}{r}A' - 2(n-3)\frac{1}{r^2}(A-1) = 0. \quad (35)$$

This equation can be exactly solved by noticing that it is written as

$$\begin{aligned}&A'' + (n-4)\frac{1}{r}A' - 2(n-3)\frac{1}{r^2}(A-1) \\ &= \frac{1}{r^{n-3}} \frac{d^2}{dr^2} [r^{n-3}(A-1)] - (n-2) \frac{1}{r^{n-2}} \frac{d}{dr} [r^{n-3}(A-1)].\end{aligned}\quad (36)$$

Hence, $A(r)$ is given by

$$A(r) = 1 - \frac{2M}{r^{n-3}} + ar^2, \quad (37)$$

where M and a are integration constants. From the boundary condition (25), we have to choose $a = 0$, and we can obtain the expression for $B(r)$ in terms of the gauge condition (32). Consequently, we arrive at the expressions for $A(r)$ and $B(r)$

$$A(r) = 1 - \frac{2M}{r^{n-3}}, \quad B(r) = \frac{2M}{r^2(r^{n-3} - 2M)}. \quad (38)$$

Then, the line element is of form

$$ds^2 = - \left(1 - \frac{2M}{r^{n-3}} \right) dt^2 + (dx^i)^2 + \frac{2M}{r^2(r^{n-3} - 2M)} (x^i dx^i)^2. \quad (39)$$

Accordingly, we have succeeded in showing that the Schwarzschild metric in the Cartesian coordinate system is a classical solution in the WTDiff gravity as in general relativity.

Here we should refer to an important remark. The Schwarzschild metric in the Cartesian coordinate system, (39) can be rewritten in the spherical coordinate system as

$$ds^2 = - \left(1 - \frac{2M}{r^{n-3}} \right) dt^2 + \frac{1}{1 - \frac{2M}{r^{n-3}}} dr^2 + r^2 d\Omega_{n-2}^2, \quad (40)$$

where

$$d\Omega_{n-2}^2 = d\theta_2^2 + \sin^2 \theta_2 d\theta_3^2 + \cdots + \prod_{i=2}^{n-2} \sin^2 \theta_i d\theta_{n-1}^2. \quad (41)$$

This form of the Schwarzschild metric is very familiar with physicists, but this is not a classical solution to the equations of motion of the WTDiff gravity, (17). The reason is that when transforming from the Cartesian coordinates to the spherical coordinates, we have a non-vanishing Jacobian factor which is against the TDiff. To put differently, while the determinant of the metric tensor in Eq. (39) is -1 , the one in Eq. (40) is not so, which is against the gauge condition (22).

4 Discussions

In this article, in order to have the WTDiff gravity, starting with the conformally invariant scalar-tensor gravity in a general space-time dimension which is invariant under both the local Weyl transformation and the diffeomorphisms (Diff), we have gauge-fixed the longitudinal diffeomorphism, by which the full diffeomorphisms (Diff) are broken to the transverse diffeomorphisms (TDiff). It is explicitly checked that not only the resultant action of the

WTDiff gravity but also its equations of motion are invariant under both the local Weyl transformation and the TDiff.

Moreover, we have studied classical solutions of the WTDiff gravity, and found that the Schwarzschild metric is certainly a solution when the metric is expressed in terms of the Cartesian coordinate system. It is of interest to note that the familiar Schwarzschild metric in the spherical coordinate system is not a classical solution in the WTDiff gravity. This dependence on the coordinate systems of the classical solutions is a general feature in the WTDiff gravity since when fixing the Weyl symmetry by a gauge condition, only the TDiff are remained in the WTDiff gravity. For instance, in the WTDiff gravity, if we rewrite a flat Minkowski space-time in the spherical coordinates, the resulting line element is not a classical solution owing to the nonvanishing Jacobian factor even if it is a solution in the Cartesian coordinates.

As a future problem, it might be possible to show that the Reissner-Nordstrom metric, which is a static solution to the Einstein-Maxwell field equations, is a classical solution in the WTDiff gravity in *four* space-time dimensions. We wish to consider this problem in near future.

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